HSE MDI: Probability Theory.
Counting principles review.

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1 Counting principles

In many situations in Probability it would be useful to have an effective method for counting the number of ways that things can occur. In fact, many problems in probability theory can be solved simply by counting the *number of different ways* that a certain event can occur. The mathematical theory of counting is formally known as combinatorial analysis.

1.1 Multiplication rule

This basic principle of counting is fundamental to all our work. In a simple representation it states that if one experiment can result in any of m possible outcomes and if another experiment can result in any of n possible outcomes, then there are mn possible outcomes of the two experiments.

Example 1.1. Random experiment consists in observing two results: one of tossing 6-sided dice and the second of taking one card out of 52. How many simple outcomes are in this experiment?

Solution. We regard tossing a dice and taking a card as results of small random experiments on their own. For each number on face side of the dice there exist 52 possibilities for the card, and *vice versa* for each card there exist 6 possibilities for the face side. So, simple by intuition there are $6 \cdot 52 = 312$ possible pairs. Which is going along with multiplication rule precisely.

We can easily generalize the basic counting principle.

If r experiments that are to be performed are such that the first one may result in any of n_1 possible outcomes; and if, for each of these n_1 possible outcomes, there are n_2 possible outcomes of the second experiment; and if, for each of the possible outcomes of the first two experiments, there are n_3 possible outcomes of the third experiment; and so on..., then there is a total of $N = n_1 \cdot n_2 \cdots n_r$ possible simple outcomes of the r experiments.

Example 1.2. A college planning committee consists of 3 freshmen, 4 sophomores, 5 juniors, and 2 seniors. A subcommittee of 4, consisting of 1 person from each class, is to be chosen. How many different subcommittees are possible?

Solution. We may regard the choice of a subcommittee as the combined outcome of the four separate small random experiments of choosing a single representative from each of the classes. It then follows from the generalized multiplication rule that there are $3 \cdot 4 \cdot 5 \cdot 2 = 120$ possible subcommittees.

Example 1.3. How many functions defined on n points are possible if each functional value is either 0 or 1?

Solution: Let the points be $1, 2, \ldots, n$. We regard value of the function at each point as a small random experiment. Since for $f(i)$ there are only two possible values for each $i = 1, 2, \ldots, n$, it follows that there are 2^n possible functions.

1.2 Permutations

How many different *ordered* collections (sequences) can you create from, say, set of elements $\{a, b, c\}$? By just enumeration we can say that there are six of them: *abc*, *acb*, *bca*, *bac*, *cab*, *cba*. It is quite intuitive: we can choose first element in one out of three ways, second element in one out of two ways, and the last element is the remaining one. This gives us $3 \cdot 2 \cdot 1 = 6$ possible variants.

We can then generalize this approach. Suppose we have set of n elements. How many different ordered collections we can construct from it?

$$n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1 = n! \text{ different permutations of the } n \text{ objects.}$$

Example 2.1. A class consists of 6 men and 4 women. An examination is given, and the students are ranked according to their performance. Assume that no two students obtain the same score.

- How many different rankings are possible?
- If the men are ranked just among themselves and the women just among themselves, how many different rankings are possible?

Solution. (a) Because each ranking corresponds to a particular ordered arrangement of the 10 people, the answer to this part is $10! = 3,628,800$. (b) Since there are $6!$ possible rankings of the men among themselves and $4!$ possible rankings of the women among themselves, it follows from the basic principle that here are $(6!)(4!) = (720)(24) = 17,280$ possible rankings in this case.

Example 2.2. Ms. Jones has 10 books that she is going to put on her bookshelf. Of these, 4 are mathematics books, 3 are chemistry books, 2 are history books, and 1 is a language book. Ms. Jones wants to arrange her books so that all the books dealing with the same subject are together on the shelf. How many different arrangements are possible?

Solution. We have 4 groups of books, each of them inside can be ordered by its own. So we have $4!$ arrangements for mathematics books, $3!$ for chemistry, and $2!$ for history books. But also we have a variability for the ordering of these groups on the shelf. To place first group we have 4 variants, then to place second group we have 3 variants, and finally we have $4!$ variants of how to place groups of books on the shelf.

As a result, in total, we have $4! \cdot 4! \cdot 3! \cdot 2! \cdot 1$ different arrangements.

1.3 Combinations

We are often interested in determining the number of different *sets* that could be formed from a total of n objects. For instance, how many different groups of 3 could be selected from the 4 items A, B, C, and D? The intuition is the following: we can choose the first object in one out 4 ways, the second in one out of 3 ways, and the third in one out 2 ways. However, using this idea from permutations, final number will include all different orderings of group of size 3, *e.g.* for group ABC there will be counted ABC, CBA, BAC, and others. To calculate number of different groups we need then to divide by number of possible permutations of 3 elements:

$$\frac{4 \cdot 3 \cdot 2}{3 \cdot 2 \cdot 1} = 12.$$

Here is the major difference between Permutations and Combinations. Order matters for Permutations, and we want to find out the total number of *ordered* collections of a given size. On the other hand, Combinations **do not** care about ordering at all, the main idea is to find out number of unique sets of r elements which can be constructed from n elements. Also I intentionally use word *set* for them, to emphasise that order makes no sense, as standard sets have no ordering.

Finally we can generalize the formula. We define this number of unique sets of r elements out of n as C_n^r, and so:

$$C_n^r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r \cdot (r-1)\cdots 2 \cdot 1} = \frac{n!}{(n-r)!r!}.$$

Example 3.1. A committee of 3 is to be formed from a group of 20 people. How many different committees are possible? Solution: We need to obtain number of unique sets of power 3 taken from 20 elements. $C_{20}^3 = \frac{20!}{17!3!} = \frac{18 \cdot 19 \cdot 20}{6} = 1140$ possible committees.

Example 3.2. From a group of 5 women and 7 men, how many different committees consisting of 2 women and 3 men can be formed? What if 2 of the men are feuding and refuse to serve on the committee together? Solution: As there are C_7^3 groups of 3 men and C_5^2 groups of 2 women, it follows from the basic principle that there are $C_7^3 C_5^2 = 350$ possible committees consisting of 2 women and 3 men. Now consider that 2 men refuse to serve together. Because a total number of $C_5^1 = 5$ groups contain these two feuding men, it means that there is $C_7^3 - 5 = 30$ groups that do not contain both of them. Because there are still $C_5^2 = 10$ ways to choose the 2 women, there are $30 \cdot 10 = 300$ possible committees in this case.

Example 3.3. An bag contains 15 marbles of which 10 are red and 5 are white. 4 marbles are selected from the bag.

1. How many (different) samples (of size 4) are possible?
2. How many samples (of size 4) consist entirely of red marbles?
3. How many samples have 2 red and 2 white marbles?
4. How many samples (of size 4) have exactly 3 red marbles?
5. How many samples (of size 4) have at least 3 red?
6. How many samples (of size 4) contain at least one red marble?