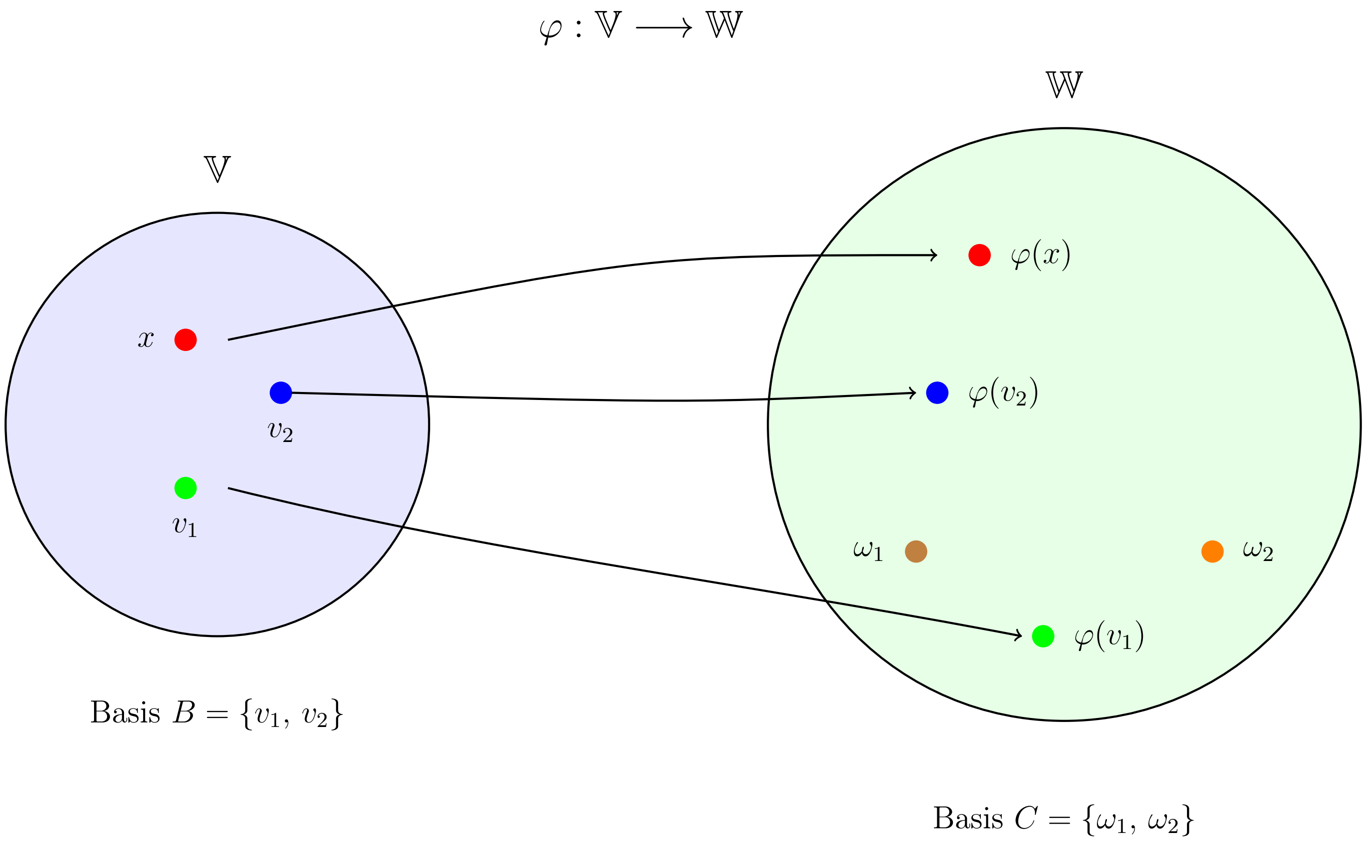
Mind Map: Linear Transformations and Their Matrix Representation



Initial assumptions:

- Let us take a look at the linear transformation $\varphi : \mathbb{V} \longrightarrow \mathbb{W}$.
- Suppose, that $\mathbb{V}$ and $\mathbb{W}$ have basis sets B and C respectively.
- For simplicity, we draw only useful for us elements of $\mathbb{V}$ and $\mathbb{W}$. Of course it could surely be infinitely many elements in each of these spaces.

Linear Transformations

$$\varphi : \mathbb{V} \to \mathbb{W}$$
$$\varphi(\alpha u + \beta v) = \alpha\varphi(u) + \beta\varphi(v)$$

Basis in $\mathbb{V}$

$$B = \{v_1, v_2\}$$
$$x = x_1 v_1 + x_2 v_2,\ [x]_B = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\varphi(x) = \varphi(x_1 v_1 + x_2 v_2) = x_1 \varphi(v_1) + x_2 \varphi(v_2)$$

Key Idea: We can compute $\varphi(x)$ for any $x \in \mathbb{V}$ if we just know the image of the basis vectors $\varphi(v_1)$ and $\varphi(v_2)$

Basis in $\mathbb{W}$

$$C = \{\omega_1, \omega_2\}$$
$$\varphi(v_1) = a_{11}\omega_1 + a_{21}\omega_2,\ [\varphi(v_1)]_C = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$$
$$\varphi(v_2) = a_{12}\omega_1 + a_{22}\omega_2,\ [\varphi(v_2)]_C = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$$

Matrix of Linear Transformation

$$\varphi(x) = x_1 (a_{11}\omega_1 + a_{21}\omega_2) + x_2 (a_{12}\omega_1 + a_{22}\omega_2) = \underbrace{(a_{11}x_1 + a_{12}x_2)}_{\gamma_1}\omega_1 + \underbrace{(a_{21}x_1 + a_{22}x_2)}_{\gamma_2}\omega_2$$

$\varphi(x) = \gamma_1\omega_1 + \gamma_2\omega_2$ - decomposition of $\varphi(x)$ in $\mathbb{W}$

$$[\varphi(x)]_C = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$[\varphi(x)]_C = A_\varphi\,[x]_B$$

- A_φ - matrix of linear transformation φ
- $[x]_B$ - coordinates of x in basis B
- $[\varphi(x)]_C$ - coordinates of $\varphi(x)$ in basis C

Key Ideas:

- Linear transformations are completely determined by their action on basis vectors
- Matrix representation allows to connect through the matrix-vector multiplication the **preimage** x and **image** $\varphi(x)$ coordinates in bases in domain $\mathbb{V}$ and target space $\mathbb{W}$ respectively