Mathematical Statistics

Class 8. Hypothesis Testing.

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Introduction

So far we have focused on the problem of estimation the unknown population parameter based on the observable data. We wanted to obtain the best possible guess $\hat{\theta} = T(X_1, \ldots, X_n)$ of the parameter θ of the distribution in population. In this case we called this problem a **point estimation**, i.e. we obtained a specific point $\hat{\theta} \in \Theta$ in the parametric space. Or we wanted to estimate the confidence interval $(L(X), U(X)) \subset \Theta$, such that it contains true parameter value with probability $(1-\alpha)$, $0 < \alpha < 1$.

Now we turn to, in some sense, simpler problem: to determine whether the unknown θ belongs to one subset $\Theta_1 \subset \Theta$ or to another $\Theta_2 \subset \Theta$.

Initial setup

Let $X_1, \ldots, X_n \tilde{f}(x|\theta)$, where θ is unknown parameter of the distribution in population. Assume that θ lies in the parameter space Θ, i.e. $\theta \in \Theta$. The hypothesis testing problem is specified by splitting the parameter space Θ into two disjoint subsets: $\Theta = \Theta_0 \cup \Theta_1$, $\Theta_0 \cap \Theta_1 =$. Then two hypotheses are stated in the following manner:

$$H_0 : \theta \in \Theta_0 \qquad H_1 : \theta \in \Theta_1,$$

where we call H_0 the *null hypothesis*, and H_1 the *alternative hypothesis*.

Our goal: to use only the given sample $X = X_1, \ldots, X_n$ to decide between H_0 and H_1. We need to construct specific test procedure, which tells us whether to reject, or do not reject H_0.

Important idea on this step is that conclusions: a) never say anything directly about H_1, and b) never say that one of two hypotheses is true. We can not conclude just from the sample data that either H_0 or H_1 is true. We may say 'accept H_0', but in fact it means 'do not reject H_0'. Practically there is no difference, but logically the difference is huge: it is impossible for an experiment to confirm a theory, but when enough evidence exists to suggests that a theory is false, it is standard to reject that theory and develop a new one. That's why sometimes H_0 is called a *conservative* hypothesis, which is not to be rejected unless the evidence is clear. On the other hand an alternative H_1 is sometimes referred to as the *research worker's* hypothesis, so it's like an competitive idea, the set of evidences which we would like to test.

There are four possible outcomes in hypothesis testing problem:

	H_0 is true	H_0 is false
Accept H_0	correct	Type II error
Reject H_0	Type I error	correct

For instance we could decide that $\theta \in \Theta_1$ when really $\theta \in \Theta_0$, that is Type I error, or we could decide that $\theta \in \Theta_0$ when, in fact, $\theta \in \Theta_1$, which is Type II error respectively. Still remember that "accept H_0 " actually means "Do not reject H_0 ."

Probability of Type I error is also often called the *significance level* of the test and denoted by α. In most problems α is selected before the test is performed and usually it is fixed at some small value, by default $\alpha = 0.05$.

Hypothesis about population mean, known population variance

1. Manufacturer of the detergent claims that the contents of boxes sold weight on average at least 16 ounces. The distribution of weights is known to be normal with standard deviation 0.4 ounce. A random sample of sixteen boxes yielded a sample mean weigh of 15.84 ounces. Test at 10% significance level the null hypothesis that the population mean weight is at least 16 ounces.

2. It is reported that the lake water contains 0.5g of salt per 1 liter, with a standard deviation 0.1g. In order to check this statement, 20 samples of water were chosen and the salt in a sample of one liter was 0.57g. Is the report of the salt content correct?

Hypothesis about population proportion

1. A concerned group of citizens wants to show that less than half of the voters supports a new law. Let p = proportion of voters, who supports it.

   - Determine H_0 and H_1.
   - If a random sample of 500 voters gives 228 in support, what does the test conclude? Use $\alpha = 0.05$. Also, evaluate p-value.

Hypothesis about population mean, unknown population variance

1. A physical model suggests that the mean temperature increase in the water used as a coolant in a compressor chamber should not be more than $5°C$. Temperature increases in the coolant measured on 8 independent runs of the compressing unit revealed the following data:

   $$6.4, 4.3, 5.7, 4.9, 6.5, 5.9, 6.4, 5.1$$

   Does the data contradict to the assertion of the physical model? (Test at $\alpha = 0.05$.) State the assumption you make about the population.

2. A company selling franchises advertises that operators obtain, on average during the, first year, a yield of 10% on their initial investment. A random sample of ten of these franchises produced the following yields for the first year of operation:

   6.1 9.2 11.5 8.6 12.1 3.9 8.4 10.1 9.4 8.9

   Assuming that population yields are normally distributed, test the company's claim.