Conditional distributions

- we have joint, marginals, it's time for a new one.

Let X, Y - R.V.

If we know that Y takes specific value y ($p_Y(y) > 0$), this gives partial knowledge of X.

Conditional PMF:

$$p_{X|Y}(x|y) = P\{X = x \mid Y = y\} = \frac{\overbrace{P\{X=x, Y=y\}}^{\text{joint}}}{\underbrace{P\{Y=y\}}_{\text{marginal}}} = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

For a fixed $Y=y$, $p_{X|Y}(x|y)$ is a valid P.M.F. for X:

- assigns nonnegative values for $\forall x$
- adds up to one: $\sum_x p_{X|Y}(x|y) = 1$

We can obtain joint pmf from conditional & marginal:

$$p_{X,Y}(x,y) = p_{X|Y}(x|y) \cdot p_Y(y)$$

Also, there is Conditional CDF of X given $Y=y$:

$$F_{X|Y}(x|y) = P\{X \leq x \mid Y=y\} = \sum_{a \leq x} p_{X|Y}(a|y)$$

If X and Y are independent: $P\{X=x \mid Y=y\} = \frac{P\{X=x\} \cdot P\{Y=y\}}{P\{Y=y\}} = P\{X=x\}$

Example:

	$Y=0$	$Y=1$
$X=0$	0.4	0.2
$X=1$	0.1	0.3

(Ross, 4a)

$P_{X|Y}(x|y=1)$ - ?

$P\{Y=1\} = 0.2+0.3 = 0.5$ $\left(\sum_x P\{X=x, Y=y\}\right)$

marginal

$$\rightarrow P\{X=0|Y=1\} = P_{X,Y}(0|1) = \frac{P\{X=0, Y=1\}}{P\{Y=1\}} = \frac{2}{5}$$

$$P\{X=1|Y=1\} = \frac{3}{5}.$$

Example: (2.12 MIT)

4 independent rolls of a 6-sided die. X - # of 1's
Y - # of 2's

Find joint PMF $P_{X,Y}\{X=x, Y=y\}$.

Idea: we need to use conditional p.m.f., because we can't find joint by two marginals (X and Y probably dependent)

$$P_X\{X=x\} = C_4^x \left(\frac{1}{6}\right)^x \cdot \left(\frac{5}{6}\right)^{4-x} \quad \text{Binomial}$$

$P_{Y|X}(y|x)$ - ? with fixed x, it should be function of y

If we have already X 1s, then there is $(4-X)$ possible situations left.

$$P\{Y=y|X=x\} = C_{4-x}^y \left(\frac{1}{5}\right)^y \cdot \left(\frac{4}{5}\right)^{4-x-y} \rightarrow \text{joint pmf:}$$

$$P\{X=x, Y=y\} = P\{Y=y|X=x\} \cdot P\{X=x\} = C_{4-x}^y \cdot C_4^x \cdot \left(\frac{1}{5}\right)^y \cdot \left(\frac{4}{5}\right)^{4-x-y} \cdot \left(\frac{1}{6}\right)^x \left(\frac{5}{6}\right)^{4-x}$$

Expectation, Variance & correlation

We can extend idea of expectation to the random vector case.

If $Z = g(X,Y)$ - function of random variables, then

$$E(Z) = \sum_{x,y} g(x,y) \cdot P_{X,Y}(x,y)$$

joint pmf

But still each of the component X, Y has its own numerical parametres.

For the linear case:

$$E[aX + bY + c] = aE[X] + bE[Y] + c$$

Special case of $Z = g(X,Y) = X \cdot Y$, or $Z = h(X) \cdot t(Y)$

If X, Y - independent:

$$E[h(X) \cdot t(Y)] = E[h(X)] \cdot E[t(Y)]$$

It is important, since:

$$\mathrm{Var}\{X+Y\} = E[\{(X+Y) - E(X+Y)\}^2] =$$

Covariance

$$= \mathrm{Var}(X) + 2E[(X - E[X]) \cdot (Y - E[Y])] + \mathrm{Var}(Y)$$

$$\mathrm{Cov}(X,Y) = E[(X - E[X]) \cdot (Y - E[Y])] = E[X \cdot Y] - E[X] \cdot E[Y]$$

if X, Y independent $\longrightarrow \mathrm{cov}(X,Y) = 0$

correlation of X, Y is <u>quantity</u> $\rho(X,Y)$:

$$\rho(X,Y) = \frac{\operatorname{cov}(X,Y)}{\sqrt{\operatorname{Var}(X)\cdot\operatorname{Var}(Y)}}$$

whenever $\operatorname{Var}(X), \operatorname{Var}(Y)$ exist and $\operatorname{Var}(X)\cdot\operatorname{Var}(Y) \neq 0$

In other words $\rho(X,Y)$ – numerical quantity, which helps to understand mutual behaviour of X, Y.

Properties

1. $-1 \le \rho(X,Y) \le 1$
2. If X, Y independent $\Rightarrow \rho(X,Y) = 0$
3. If $\rho = 1\ (-1) \Rightarrow Y = \alpha X + \beta$, $\alpha > 0\ (\alpha < 0)$

Ex. 5.1.6

	$Y=0$	$Y=1$	
$X=0$	$1/2$	0	$1/2$
$X=1$	$1/3$	$1/6$	$1/2$
	$5/6$	$1/6$	

$E[X] = \{\text{use marginals}\} = 1/2 \cdot 0 + 1/2 \cdot 1 = 1/2$

$E[Y] = 1/6$

$\operatorname{Var}(X) = E[X^2] - (E[X])^2 =$

$= 1/2 - 1/4 = 1/4$

$\operatorname{Var}(Y) = 1/6 - 1/36 = \frac{5}{36}$

$\operatorname{Cov}(X,Y) = \frac{1}{6} - \frac{1}{12} = \frac{1}{12}$

$E[X\cdot Y] = 1/6$

$$\rho(X,Y) = \frac{1/6 - \frac{1}{12}}{\sqrt{\frac{5}{36}\cdot 1/4}}$$