$\hat{\Theta} = T(X_1, \dots, X_n)$

our guess

$\mathbb{E}[\hat{\Theta}] = \Theta$

unbiased

$T = T(X_1 \dots X_n)$

R.V.

$T$ has its own distribution

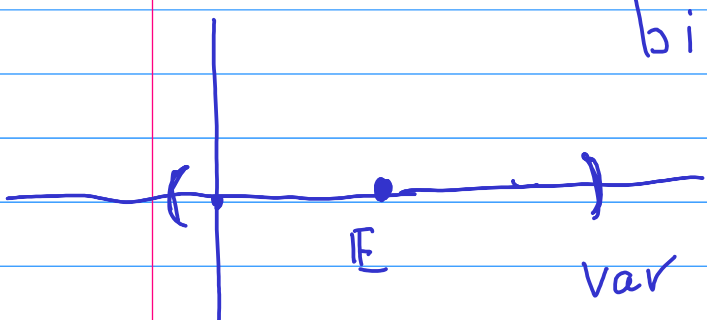
↓

$T$ has $\mathbb{E}$, Var, ...

$$\text{MSE} = \mathbb{E}(\hat{\Theta} - \Theta)^2 =$$

$$= \text{Var}(\hat{\Theta}) + (\text{bias})^2$$

$$\text{bias} = \mathbb{E}[\hat{\Theta}] - \Theta$$



$P_X(x)$

1.

| $X$ | $0$ | $1$ |
|---|---|---|
| $p$ | $\frac{1}{2}$ | $\frac{1}{2}$ |

1. $\mu$, $\sigma^2$

$$\mu = \mathbb{E}(X) = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2}$$

$$\sigma^2(X) = \mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 =$$

$$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$P_X(x)$

2. $X_1, X_2, \dots, X_9$

i.i.d.

distr. $F(x)$ $\to$ $X$

$$\hat{\mu}_2 = X_1$$

$$\mathbb{E}[\hat{\mu}_2] = \mathbb{E}[X_1] = \boxed{\mu} \ !!!$$

$\hat{\mu}_2$ - unbiased.

$\hat{\mu}_3 = \bar{X}$ $\qquad$ $\mathbb{E}[\bar{X}] = \mu$

unbiased

$$\bar{X} = \frac{X_1 + \dots X_9}{9} = \frac{\sum_{i=1}^{9} \mathbb{E}[X_i]}{9} =$$

$$= \frac{9 \cdot \mu}{9} = \mu.$$

$$\hat{\mu}_4 = X_1 + \frac{1}{3} X_2$$

$$\mathbb{E}[\hat{\mu}_4] = \mathbb{E}\left(X_1 + \frac{1}{3} X_2\right) = \mathbb{E}[X_1] + \frac{1}{3}\mathbb{E}[X_2] =$$

$$= \mu + \frac{1}{3} \cdot \mu = \frac{4}{3} \cdot \mu$$

$\hat{\mu}_4$ has bias $= \frac{1}{3} \cdot \mu$

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$MSE(\hat{\mu}_2)$ : $\qquad$ $bias(\hat{\mu}_2) = 0$

$Var[X_1] = \sigma^2$ $\qquad$ $MSE = var(\hat{\mu}_2) + bias(\hat{\mu}_2) =$

$$= \boxed{\sigma^2}$$

$$\hat{\mu}_3 = \overline{X} \qquad \text{bias}(\hat{\mu}_3) = 0$$

$$Var(\hat{\mu}_3) = \frac{\sigma^2}{n} = \frac{\sigma^2}{9}$$

$$MSE(\hat{\mu}_3) = \frac{\sigma^2}{9} + 0$$

$$\hat{\mu}_4 = X_1 + \frac{1}{3} X_2$$

$$Var(\hat{\mu}_4) = var(X_1) + \frac{1}{9} Var(X_2) =$$

$$= \sigma^2 + \frac{1}{9}\sigma^2 = \frac{10}{9}\sigma^2$$

$$MSE(\hat{\mu}_4) = \frac{10}{9}\sigma^2 + \frac{1}{9}\mu^2$$

(2) $X_1, X_2, X_3$

$\mu, \sigma^2$

$\hat{\sigma}^2 = c \cdot (X_1 - X_2)^2$ ; we hope that is a good guess for variance

$\mathbb{E}[\hat{\sigma}^2] = \sigma^2$ $\qquad \hat{\sigma}^2 = \hat{\theta}$ ; $\sigma^2 = \theta$

$\hat{\theta} = cX_1^2 - 2cX_1X_2 + cX_2^2$

$\mathbb{E}[\hat{\theta}] = c \cdot \mathbb{E}[X_1^2] - 2c\,\mathbb{E}[X_1] \cdot \mathbb{E}[X_2] + c\,\mathbb{E}[X_2^2]$

$\mathbb{E}[X^2] = \mathrm{Var}[X] + (\mathbb{E}[X])^2$

$\mathbb{E}[\hat{\theta}] = c \cdot (\sigma^2 + \mu^2) - 2c(\mu \cdot \mu) + c(\sigma^2 + \mu^2)$

$= c \cdot \sigma^2 + \cancel{c\mu^2} - \cancel{2c\mu^2} + c\sigma^2 + \cancel{c\mu^2} = 2c\sigma^2 = \sigma^2 \rightarrow c = \frac{1}{2}$

(3) $X_1$, $X_2$

$\overline{X} = \dfrac{X_1 + X_2}{2}$ ; $Y = \dfrac{1}{3} X_1 + \dfrac{2}{3} X_2$

$\mathbb{E}[\overline{X}] = \mu$

$\uparrow$ unbiased

$\mathbb{E}[Y] = \frac{1}{3} \cdot \mathbb{E}[X_1] + \frac{2}{3} \mathbb{E}[X_2] =$

$= \frac{1}{3} \cdot \mu + \frac{2}{3} \cdot \mu = \mu$

$\uparrow$ unbiased

$\mathrm{Var}[\overline{X}] = \dfrac{\sigma^2}{n}$

$MSE[\overline{X}] = \dfrac{\sigma^2}{n} = \dfrac{\sigma^2}{2}$

$\mathrm{Var}(Y) = \frac{1}{9} \mathrm{Var}(X_1) +$

$+ \frac{4}{9} \mathrm{Var}(X_2) =$

$= \frac{5}{9} \cdot \sigma^2$

$MSE[Y] = \frac{5}{9} \sigma^2$

$MSE[\overline{X}] < MSE[Y]$