$$P_{X|Y}(x,y) = \frac{P_{X,Y}(x,y)}{P_Y(y)}$$

$$\boxed{P_{X,Y}(x,y) = P_{X|Y} \cdot P_Y(y)}$$

Ex. 4 rolls of 6-sided dice

X - # of 1's

Y - # of 2's

$P_{X,Y}\{X=x, Y=y\}$ - ?

$0 \leq X+Y \leq 4$

$F_X \cdot F_Y$

$P_X \cdot P_Y = P_{X,Y}$

$0 \quad 0 \quad \underbrace{1 \quad 1}$

$P_X\{X=x\}$

$\overset{4-Y}{Y}$ # of 2's

X - # of 1s

$$P_X\{X=x\} = C_4^x \left(\frac{1}{6}\right)^x \left(\frac{5}{6}\right)^{4-x}$$

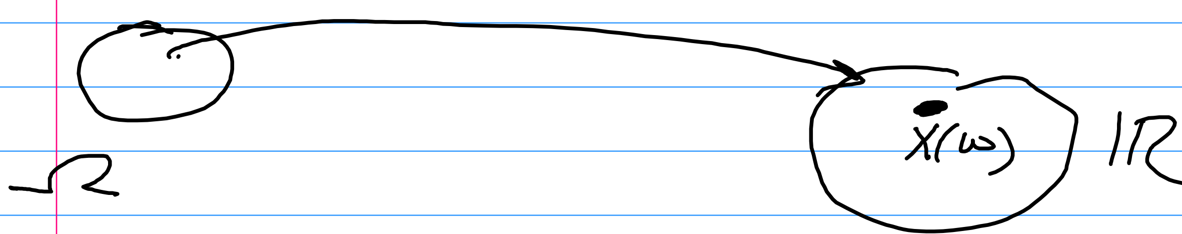
$$P_{Y|X}\{Y=y \mid X=x\} =$$

$$= C_{4-x}^y \left(\frac{1}{5}\right)^y \left(\frac{4}{5}\right)^{4-x-y}$$

$$\boxed{P_{X,Y}} = \underbrace{P_{Y|X} \cdot P_X\{X=x\}}_{0 \leq X+Y \leq 4}$$

$P_{X,Y} = 0$ otherwise.

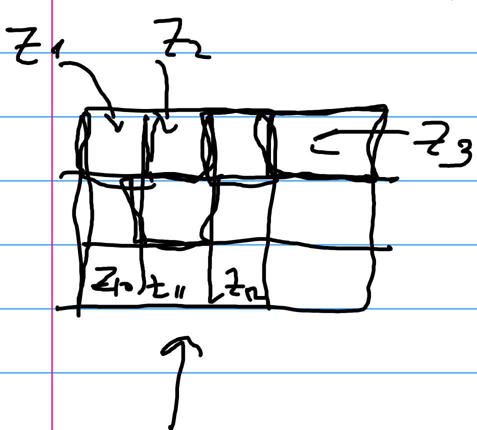
$$E[X] = \sum_{\Omega_X} x \cdot P\{X=x\}$$

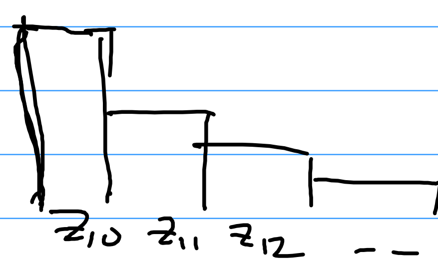


$$E[X,Y] = (E[X], E[Y])$$

$$Z = g(X,Y) \text{ — 1 number}$$

$$E[Z] = \sum_{x,y} g(x,y) \cdot P\{X=x, Y=y\}$$





$$Z = X \cdot Y$$

$$E[X \cdot Y] = \sum_{x,y} x \cdot y \cdot P\{X = x, Y = y\}$$

$$Var(X+Y) = Var(X) + Var(Y) + 2 \cdot E[(X - E[X])(Y - E[Y])]$$

$$Cov(X, Y) = E[(X - E[X]) \cdot (Y - E[Y])] =$$

$$= E[X \cdot Y] - E[X] \cdot E[Y]$$

X, Y - independent:

$$E[X \cdot Y] = E[X] \cdot E[Y]; \quad Cov(X,Y) = 0$$

$E[X] = \sum_{x} x \cdot P\{X = x\}$ marginal p.m.f.

Correlation

$$\rho(X,Y) = \frac{\operatorname{cov}(X,Y)}{\sqrt{\operatorname{Var}(X)\cdot\operatorname{Var}(Y)}}$$

1. $-1 \le \rho(X,Y) \le 1$.

2. X, Y indep. $\Rightarrow \rho(X,Y) = 0$

3. $\rho = 1$ $\qquad Y = \alpha \cdot X + \beta;\ \alpha > 0$

   $\rho = -1$ $\qquad Y = \alpha X + \beta;\ \alpha < 0$

Ex.

	$Y=0$	$Y=1$
$X=0$	$1/2$	0
$X=1$	$1/3$	$1/6$

cov - ?

ρ - ?

X	0	1
$P\{X=x\}$	$1/2$	$1/2$

Y	0	1
$P\{Y=y\}$	$5/6$	$1/6$

$$P\{X=x\} = \sum_{y} P\{X=x; Y=y\}$$

$$E[X\cdot Y] = \sum_{x,y} \underbrace{x\cdot y}_{g(x,y)} \cdot P\{X=x; Y=y\} = \frac{1}{6}$$

$=$

$$g = x\cdot y$$

$$= 0\cdot 0\cdot P + 1\cdot 0\cdot P + 0\cdot 1\cdot P +$$

$$+ 1\cdot 1\cdot \underline{P\{X=1; Y=1\}} = \frac{1}{6}$$

$$E[X] = \frac{1}{2} \quad ; \quad E[Y] = \frac{1}{6}$$

$$E[X] = \sum_{x} x\cdot P\{X=x\}$$

marginal

$$Cov(X,Y) = \frac{1}{6} - \frac{1}{12} = \frac{1}{12}$$

$$Var(X) = E[X^2] - (E[X])^2$$

$$E[X^2] = \sum x^2 \cdot P\{X=x\} = \frac{1}{2}$$

$$Var(X) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$E[Y^2] = \frac{1}{6}$$

$$Var(Y) = \frac{1}{6} - \frac{1}{36} = \frac{5}{36}$$

$$\rho = \frac{1}{12\sqrt{\frac{5}{4 \cdot 36}}} = \frac{2 \cdot 6}{12 \cdot \sqrt{5}} = \frac{1}{\sqrt{5}}$$

Independence

$$F_{X,Y} = F_X F_Y$$

marginal pmfs

D.R.V.

$$P\{X=x; Y=y\} = P\{X=x\} \cdot P\{Y=y\} \quad \forall x, y$$

C.R.V.

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y) \quad \forall x, y$$

joint pdf

marginal pdfs

$$\underbrace{P\{Y>X\}}$$

	$Y=0$	1	2	3
$X=-1$	~~0.02~~	~~0.03~~	~~0.09~~	~~0.01~~
0	0.04	~~0.2~~	~~0.16~~	~~0.1~~
1	0.05	0.1	~~0.15~~	~~0.05~~

$$\{Y>X\} = \sum_{Y>X} P\{X=x; Y=y\}$$

$$P\{a<X<b;\ c<Y<d\}$$

1. $P\{X=x\}$ $\quad x = 0, 1, 2, 3$

y	0	1	2	3
$P\{Y=y\}$	0.11	0.33	0.4	0.16

$$P\{X=x\} = \sum_y P\{X=x; Y=y\}$$

X	-1	0	1
$P\{X=x\}$	0.15	0.5	0.35

$$P_{X|Y}\{X=x \mid Y=y\} = \frac{P\{X=x; Y=y\}}{P\{Y=y\}}$$

X	-1	0	1
$P\{X=x \mid Y=2\}$	0.225	0.4	0.375

$1 - 0.4 - 0.225$

$\frac{16}{40} = \frac{4}{10} = 0.4$

$\frac{9}{25} = 0.225$

$\frac{0.15}{0.4} =$

$P\{Y=y \mid X=1\}$

$$E[X \cdot Y] = \sum_{x=-1}^{1} \sum_{y=0}^{3} x \cdot y \cdot P\{X=x; Y=y\}$$

$E[X] = -\cancel{1} \cdot 0.15 + 0 + 1 \cdot 0.35$

$E[Y] = 0 + 1 \cdot 0,33 + 2 \cdot 0.4 + 3 \cdot 0.16$

$$\operatorname{cov}(x,y) = E[X \cdot Y] - E[X] \cdot E[Y]$$

$$\rho = \frac{\operatorname{cov}}{\sqrt{\operatorname{var}(x)\operatorname{var}(y)}}$$