Consultation 1.

1.1.

$$\frac{\bar{X}-\mu}{\frac{S}{\sqrt{n}}} \sim N(0,1)$$

$t_{\beta/2}$ $\quad t_{\alpha/2} = 2.306$

$$\frac{E_{t,\alpha}}{E_{t,\beta}} = \frac{t_{\alpha/2}}{t_{\beta/2}}$$

$t_{\beta/2} = 3.355$

or $\quad n > 100$

$$Z = \frac{\bar{X}-\mu}{\frac{S}{\sqrt{n}}}$$

$E = t_{\beta/2} \cdot \frac{S}{\sqrt{n}} \Rightarrow$

$t_{\alpha/2} \cdot S/\sqrt{n} = m$

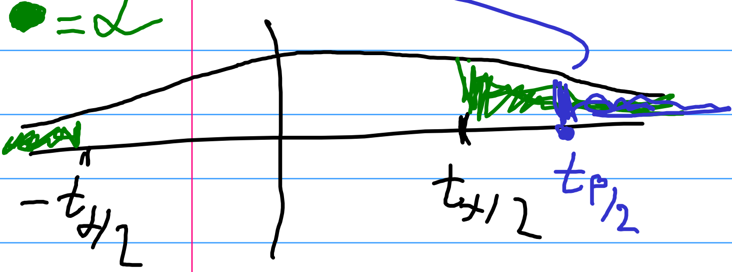
$$t(n\, d.f.) \xrightarrow[n \to \infty]{} N(0,1)$$

$$\mu \in (\bar{X} - E_t \,;\, \bar{X} + E_t)$$

$$E_t = t_{\alpha/2,\, n-1} \cdot \frac{S}{\sqrt{n}} = m \,; \quad \alpha = 0.05$$

$$\alpha/2 = 0.025$$

● $= \alpha$



$\beta = 0.01$

$\beta/2 = 0.005$ $\quad t_{\beta/2}$

$$E_{t,\beta} = \frac{m}{t_{\alpha/2}} \cdot t_{\alpha/2} = m \cdot \frac{3.355}{2.306} =$$

$$\frac{S}{\sqrt{n}}$$

$$\boxed{= 1.45 \cdot m}$$

1.2 $(1-\alpha) \cdot 100\%$

$\alpha = 0.01$ $\quad \alpha/2 = 0.005$

$$E_t = t_{\alpha/2} \cdot \frac{S}{\sqrt{n}} = t_{0.005} \frac{26}{\sqrt{10}}$$

1.3. $n = 10$ $\quad X_1 \ldots X_{10} \sim N(0.1;\ 1)$

$P(\overline{X} < 0)$; $\quad \dfrac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$

$$\overline{X} < 0 \rightarrow \frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}} < \frac{0 - \mu}{\frac{\sigma}{\sqrt{n}}}$$

$$\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} = Z < -0.1 \cdot \sqrt{10}$$

$$Z < -0.316$$

$$P(Z < -0.316) = 0.3745$$

$$-0.32$$

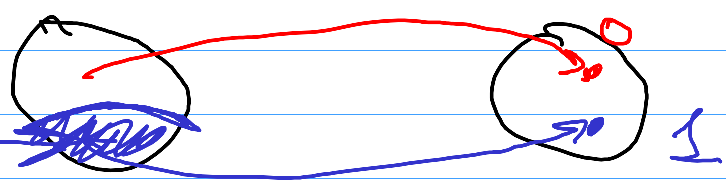
1.4. $\mathbb{E}(XY) = 2 \cdot 0{,}1 + 4 \cdot 0.3 + 4 \cdot 0.2 + 8 \cdot 0.2 + 6 \cdot 0{,}1 + 12 \cdot 0{,}1 = 2{,}2 + 1{,}6 + 0{,}6 + 1{,}2 = 5.6$

$\mathbb{E}(g(X,Y)) =$

$\rightarrow \sum g(x,y) \cdot P\{X=x;\ Y=y\}$

$\rightarrow \iint g(x,y) f_{X,Y}(x,y)\, dx\, dy$

1.5 $T(X)$



Critical region

set of all ever samples

$X : T(X) = 1$

H_0 is true.

Common: reject if $\bar{X} > c$

X $\qquad\qquad$ Y

$n = 5$ $\qquad\qquad$ $m = 5$

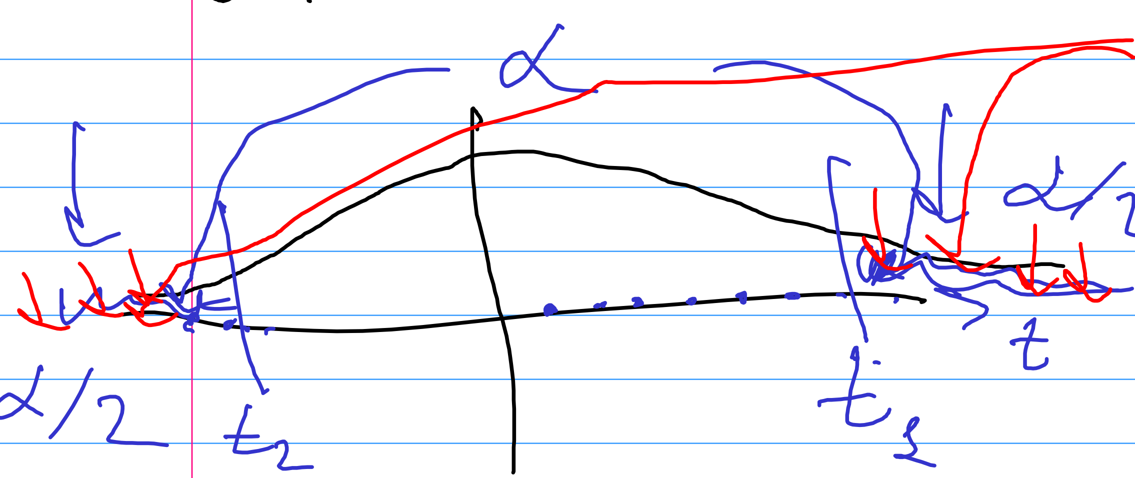
$\sigma_1 = \sigma_2 = \sigma$, unknown

$H_0: \mu_1 = \mu_2$; $H_1: \mu_1 \neq \mu_2$

$\mu_1 > \mu_2$; $\mu_1 < \mu_2$

$$t = \frac{\bar{X} - \bar{Y} - \cancelto{0}{(\mu_1 - \mu_2)}}{S \sqrt{\frac{1}{n} + \frac{1}{m}}} \sim \underbrace{t(n+m-2 \text{ d.F.})}$$

$t > c \rightarrow$ reject



$\boxed{|t| \geq 2.306}$

$t_{\alpha/2}$; 8 d.F.

$t_{\alpha/2}: P\{t \in (t_{\alpha/2}; +\infty)\} = \alpha/2$

C.R. : $X: (x_1 \ldots x_5), (Y_1 \ldots Y_5)$.

$|t| > 2.306$

We also specify critical region

specify sample

Statistic is a function from sample

2.1.

	$Y=0$	$Y=2$	$Y=4$
$X=0$	0.1	0.1	0
$X=2$	0.1	0.4	0.1
$X=4$	0	0.1	0.1

$$P\{X=x\} = \sum_{y} P\{X=x;\ Y=y\}$$

$P\{X=0\} = 0.2$; $P\{X=4\} = 0.2$

$P\{X=2\} = 0.6$

$$\sum_{\forall x} P\{X=x\} = 1$$

$P\{Y=0\} = 0.2$; $P\{Y=2\} = 0.6$

$P\{Y=4\} = 0.2$

$\forall x; y$

$$P\{X=x;\ Y=y\} = P\{X=x\} \cdot P\{Y=y\}$$

$P\{X=0;\ Y=0\} \neq P\{X=0\} \cdot P\{Y=0\} \rightarrow$ dependent

$$E(Y) = \sum_{\forall Y} y \cdot P\{Y = y\} = 2 \cdot 0.6 + 4 \cdot 0.2 = 2$$

$\uparrow$ marginal

$$E(X) = 2$$

$$\mathrm{var}(X) = E(X^2) - (E(X))^2 =$$

$$= 4 \cdot 0.6 + 16 \cdot 0.2 - 4 =$$

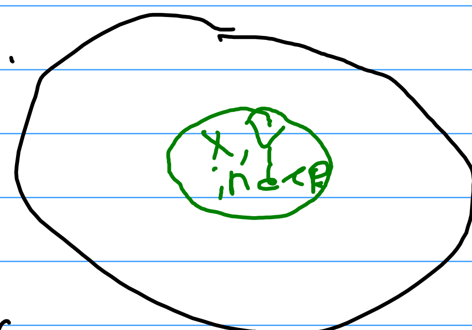
$$= 2.4 + 3.2 - 4 = 1.6$$

$$\mathrm{var}(Y) = 1.6$$

$$\mathrm{cov}(X,Y) = E(XY) - E(X) \cdot E(Y)$$

X, Y:

$\mathrm{cov}(X,Y) = 0$



$$E[g(X,Y)] = \sum g(x,y) \cdot P\{X = x; Y = y\}$$

$$= E[XY] = \sum = 4 \cdot 0.4 + 8 \cdot 0.1 + 8 \cdot 0.1 + 16 \cdot 0.1$$

$$1.6 + 1.6 + 1.6 = \cancel{4.2} \to 4.8$$

$$\mathrm{cov}(X,Y) = 4.8 - 4 = 0.8$$

$$\rho = \frac{\text{Cov}}{\sqrt{\text{var}(X)\cdot\text{var}(Y)}} = \frac{0.8}{\sqrt{1.6\cdot 1.6}} = \approx$$

$$E[g(x,y)] = \sum g(x,y)\cdot P\{X=x; Y=y\} =$$

$$= 8\cdot 0.4 + 20\cdot 0{,}1 + 20\cdot 0{,}1 + 32\cdot 0{,}1 =$$

$$g(x,y) = x^2 + y^2$$

$$= 3{,}2 + 4 + 3{,}2 = 7{,}2 + 4 = 11{,}2$$

$$E(4X + 2Y) = 4E[X] + 2E[Y] =$$

$$= 8 + 4 = \underline{12}.$$

2.2. $f_{x,y}(x,y)$: $\quad f(x,y) \geq 0$

$$\int_{-\infty}^{+\infty}\int f_{x,y}(x,y)\,dx\,dy = 1.$$

$$\int_0^1\int_0^1 (C + 0.1xy)\,dx\,dy = 1$$

$$\int_0^1 dy \left(Cx + 0.1y\frac{x^2}{2}\right)\Big|_{x=0}^{x=1} =$$

$$= \int_0^1 dy \cdot \left(C + \frac{1}{20}y\right) = Cy + \frac{1}{40}y^2\Big|_0^1 =$$

$$= C + \frac{1}{40} = 1 \rightarrow \boxed{C = \frac{39}{40}}$$

marginal pdf

$$f_X(x) = \int_{-\infty}^{+\infty} F_{X,Y}(x,y)\,dy =$$

$$= \int_0^1 \left(\frac{39}{40} + 0.1xy\right) dy =$$

$$= \frac{39}{40} y \Big|_0^1 + \frac{1}{20} x\, y^2 \Big|_{y=0}^{y=1} =$$

$$= \frac{39}{40} + \frac{1}{20} x$$

marginal pdf

$$f_Y(y) = \frac{39}{40} + \frac{1}{20} y$$

For independence:

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y) \quad \forall x,y$$

X, Y are not independent

E[X]: $\boxed{\int_0^1 x\, f_X(x)\, dx}$ $\quad E[X] = \int_{-\infty}^{+\infty} x \cdot f_X(x)\, dx$

$$F(x) = \int_{-\infty}^{x} f_X(t)\,dt$$

$$F(x) = \int_{0}^{x} \left(\frac{39}{40} + \frac{1}{20}t\right) dt =$$

$$= \frac{39}{40} \cdot x + \frac{1}{40} \cdot x^2$$

$$F(x) = P(X \leq x)$$

$x \to \infty$

$F(x) = 1$

$x \to -\infty$

$F(x) = 0$

$$F(x) = \frac{39}{40}x + \frac{1}{40}x^2, \quad 0 < x < 1$$

$$1, \quad x \geq 1$$

$$0; \quad x \leq 0$$

2.3 $\bar{X} = 5200$

$s = 150$

$n = 9$

$X_1 \ldots X_9 \sim N(\mu, \sigma^2)$

$$\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$$

σ is unknown

$$t = \frac{Z}{\sqrt{\frac{\chi^2(k)}{k}}}$$

$$\frac{s^2(n-1)}{\sigma^2} \sim \chi^2(n-1 \text{ d.f.})$$

$$t = \frac{\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}}{\sqrt{\frac{s^2(n-1)}{\sigma^2} \cdot (n-1)}} = \frac{\bar{X} - \mu}{\frac{s}{\sqrt{n}}} \sim t(n-1)$$

Z

χ^2

d.f.

CI motivation:

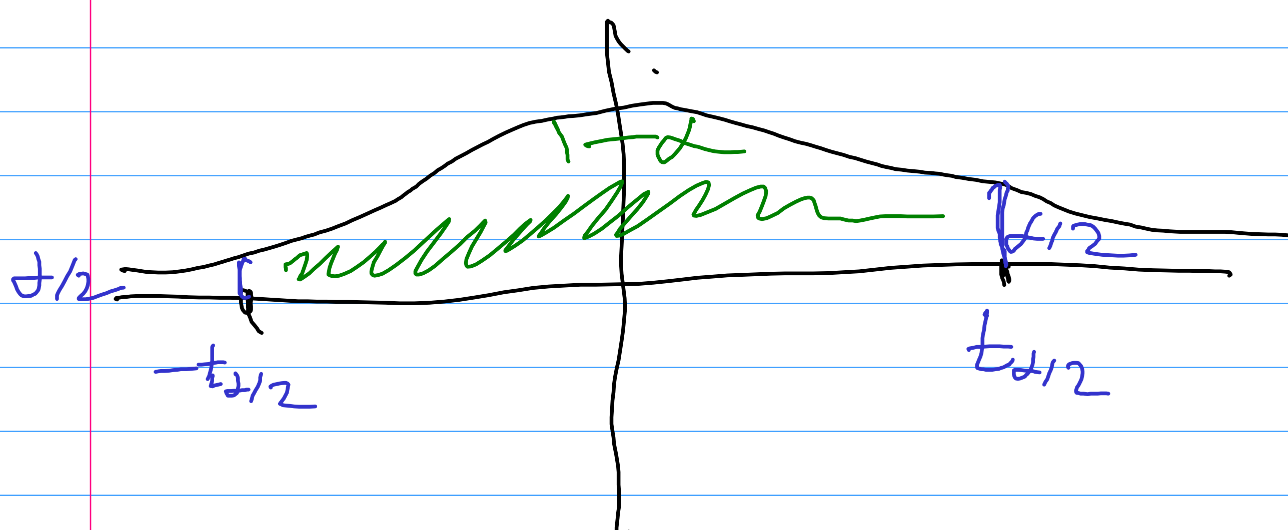
obtain Lower and Upper bound:

$$P\{\theta \in (L, U)\} = 1-\alpha$$

In our case:

t_{Lower}, $t_{upp.}$;

$$P\{t \in (t_L, t_u)\} = 1-\alpha$$



Assume we found $t_{\alpha/2}$:

$$\frac{\bar{X}-\mu}{\frac{S}{\sqrt{n}}} < t_{\alpha/2} \Rightarrow \bar{X} - t_{\alpha/2} \cdot \frac{S}{\sqrt{n}} < \mu$$

$$\frac{\bar{X}-\mu}{\frac{S}{\sqrt{n}}} > t_{\alpha/2}$$

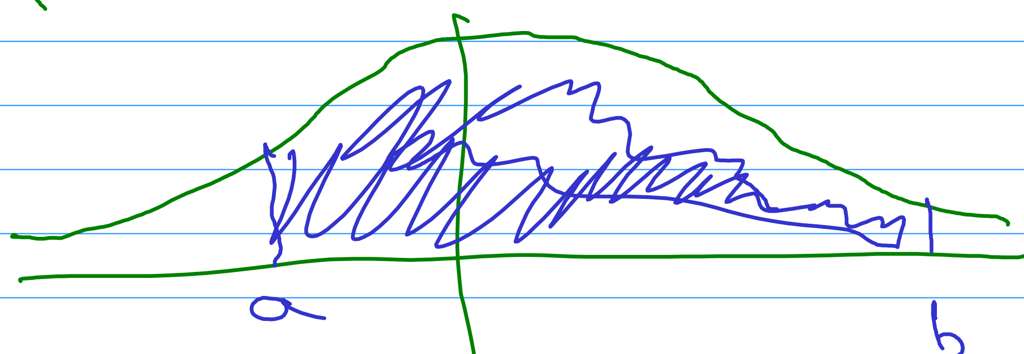
Error margin

$$\mu < \bar{X} + t_{\alpha/2} \cdot \frac{S}{\sqrt{n}}$$

$1-\alpha$

Confidence level

CI: $(a; b)$



$P\{X \in (a,b)\}$

$\alpha = 0.1$; $\alpha/2 = 0.05$

$t_{0.05} = 1.86$

$\mu \in \left(5200 - 1.86 \cdot \frac{150}{9} ; 5200 + 1.86 \frac{150}{9}\right.$